MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

VIRGINIA TECH

Balanced Truncation for bilinear systems with quadratic output

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Outline

- Part I: Working towards BQO theory
  1. BQO definition and motivation
  2. Definition of BQO Kernels
  3. Definition of Gramians
- Part II: Model order reduction for BQO systems
  1. Lifting to bilinear systems
  2. Square root balanced truncation algorithm
  3. Truncated Gramians
- Part III: Numerical results
  1. Heat transfer model (singular values)
  2. Heat transfer model (time evolution)
  3. Heat transfer model (absolute error)
- Conclusion and outlook

Part I: Working towards BQO theory

BQO definition and motivation

- A bilinear time-invariant system with quadratic outputs abbreviated as BQO, $\mathbf{\Sigma} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{N}, \mathbf{M})$ is a dynamical system given by the following set of differential equations:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{x}^\top(t)\mathbf{M}\mathbf{x}(t). \end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{y}(t) \in \mathbb{R}$ and $\mathbf{u}(t) \in \mathbb{R}$.

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- When does one encounter such systems?
  1. Studying "energies" of a bilinear system.
  2. Some bilinear port Hamiltonian systems occur as BQO systems. For example, let $\mathbf{H}(x) = \frac{1}{2}\mathbf{x}^\top\mathbf{Q}\mathbf{x}, \mathbf{T}(x) = \mathbf{N}\mathbf{x}$. Then the bilinear port Hamiltonian system,

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J} - \mathbf{R})\nabla\mathbf{H}(x) + \mathbf{T}(x)\mathbf{u} \\ \mathbf{y} &= \mathbf{T}(x)^\top\mathbf{Q}\mathbf{x} \end{aligned}$$

     is actually a BQO system!

Some key observations

Bilinear with quadratic output (BQO)	Bilinear (linear output)
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t),$ $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{x}^\top(t)\mathbf{M}\mathbf{x}(t).$	$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t),$ $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t).$
$\Sigma^{\mathsf{BQO}} \equiv (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{N}, \mathbf{M})$	$\Sigma^{\mathsf{bil}} \equiv (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{N}, \mathbf{0})$

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- Notice that the state equations and solutions for $\mathbf{x}(t)$ are the same for BQO and bilinear systems.
- As a result, the reachability Gramian for the BQO system is the same as that for the corresponding bilinear system (BQO system with $\mathbf{M} = 0$).

BQO Kernels - I

- Recall the BQO state equation given by, $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t)$

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- Let $\mathbf{x}(t) = \sum_{k=1}^{\infty} \mathbf{x}_k(t)$ with the equations for $\mathbf{x}_k$ given below, (see [Rugh, 1981])

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\dot{\mathbf{x}}_1(t) &= \mathbf{A}\mathbf{x}_1(t) + \mathbf{B}\mathbf{u}(t) \\
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- Solving recursively gives us,

$$\begin{aligned}\mathbf{x}_1(t) &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\\ \mathbf{x}_k(t) &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau \quad \text{for } (k > 1)\end{aligned}$$

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- Summing up the $\mathbf{x}_k$ terms we get,

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \sum_{k=2}^{\infty} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau$$

BQO Kernels - II

- Using $\mathbf{x}(t) = \sum_{i=1}^{\infty} \mathbf{x}_i(t)$, the expression $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{x}^\top(t)\mathbf{M}\mathbf{x}(t)$ becomes,

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\begin{aligned}
\mathbf{y}(t) &= \mathbf{C}\sum_{k=1}^{\infty} \mathbf{x}_k(t) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{x}_i(t)^\top \mathbf{M}\mathbf{x}_j(t) \\
&= \int_0^\infty \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{B}}_{\mathbf{K}_1(t)} \mathbf{U}_1(t)dt + \int_0^\infty \underbrace{\mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{B}}_{\mathbf{K}_2(t_1,t_2)} \mathbf{U}_2(t)dt + \dots \\
&+ \int_0^\infty \underbrace{\mathbf{B}^\top e^{\mathbf{A}^\top t_1}\mathbf{M}e^{\mathbf{A}t_2}\mathbf{B}}_{\mathbf{K}_{1,1}(t_1,t_2)} \mathbf{U}_{1,1}(t)dt + \int_0^\infty \underbrace{\mathbf{B}^\top e^{\mathbf{A}^\top t_1}\mathbf{M}e^{\mathbf{A}t_2}\mathbf{N}e^{\mathbf{A}t_3}\mathbf{B}}_{\mathbf{K}_{1,2}(t_1,t_2,t_3)} \mathbf{U}_{1,2}(t)dt + \dots \\
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\end{aligned}$$

- The general expression for $\mathbf{K}_{i,j}$ and $\mathbf{K}_k$ is given as,

$$\mathbf{K}_{i,j} = \mathbf{B}^\top e^{\mathbf{A}^\top t_1} \underbrace{\mathbf{N}^\top \dots \mathbf{N}^\top}_{(i-1)\ \mathbf{N}^\top \text{ terms}} e^{\mathbf{A}^\top t_i}\mathbf{M}e^{\mathbf{A}t_{i+1}} \underbrace{\mathbf{N}\dots\mathbf{N}}_{(j-1)\ \mathbf{N} \text{ terms}} e^{\mathbf{A}t_{i+j}}\mathbf{B}$$

$$\mathbf{K}_k = \mathbf{C}e^{\mathbf{A}t_1} \underbrace{\mathbf{N}\dots\mathbf{N}}_{(k-1)\ \mathbf{N} \text{ terms}} e^{\mathbf{A}t_k}\mathbf{B}$$

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\end{aligned} \tag{2}$$

- sanity check: Set $\mathbf{M} = 0$ (bilinear case)

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- **The components can be labelled with indices i, j running from $(1, 2, \ldots, \infty)$**

BQO Gramian components

Definition [P./Gosea/Pontes Duff/Benner '24]

Let $\mathbf{Q}^{\mathsf{bil}}$ be the bilinear Gramian which satisfies

$$\mathbf{A}^\top \mathbf{Q}^{\mathsf{bil}} + \mathbf{Q}^{\mathsf{bil}} \mathbf{A} + \mathbf{N}^\top \mathbf{Q}^{\mathsf{bil}} \mathbf{N} + \mathbf{C}^\top \mathbf{C} = 0$$

and $\mathbf{Q}_{i,j}$ be defined as

$$\begin{aligned}
\hat{\mathbf{P}}_k(t_1, \ldots, t_k) &= e^{\mathbf{A} t_k} \mathbf{N} \hat{\mathbf{P}}_{k-1} \text{ where } \hat{\mathbf{P}}_1(t_1) = e^{\mathbf{A} t_1} \mathbf{B}, \\
\hat{\mathbf{Q}}_{1,j}(t_1, \ldots, t_{j+1}) &= e^{\mathbf{A}^\top t_{j+1}} \mathbf{M} \hat{\mathbf{P}}_j \\
\hat{\mathbf{Q}}_{i,j}(t_1, \ldots, t_{i+j}) &= e^{\mathbf{A}^\top t_{i+j}} \mathbf{N}^\top \hat{\mathbf{Q}}_{i-1,j} \quad (\text{for } i > 1) \\
\mathbf{Q}_{i,j} &= \int_0^\infty \cdots \int_0^\infty \hat{\mathbf{Q}}_{i,j} \hat{\mathbf{Q}}_{i,j}^\top dt_1 \ldots dt_{i+j}
\end{aligned}$$

Then the observability Gramian $\mathbf{Q}$ is given by $\mathbf{Q} = \mathbf{Q}^{\mathsf{bil}} + \sum_{i=1}^\infty \sum_{j=1}^\infty \mathbf{Q}_{i,j}$.

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Let $\mathbf{Q}^{\mathsf{bil}}$ be the bilinear Gramian which satisfies

$$\mathbf{A}^\top \mathbf{Q}^{\mathsf{bil}} + \mathbf{Q}^{\mathsf{bil}} \mathbf{A} + \mathbf{N}^\top \mathbf{Q}^{\mathsf{bil}} \mathbf{N} + \mathbf{C}^\top \mathbf{C} = 0$$

and $\mathbf{Q}_{i,j}$ be defined as

$$\begin{aligned}
\hat{\mathbf{P}}_k(t_1, \ldots, t_k) &= e^{\mathbf{A} t_k} \mathbf{N} \hat{\mathbf{P}}_{k-1} \text{ where } \hat{\mathbf{P}}_1(t_1) = e^{\mathbf{A} t_1} \mathbf{B}, \\
\hat{\mathbf{Q}}_{1,j}(t_1, \ldots, t_{j+1}) &= e^{\mathbf{A}^\top t_{j+1}} \mathbf{M} \hat{\mathbf{P}}_j \\
\hat{\mathbf{Q}}_{i,j}(t_1, \ldots, t_{i+j}) &= e^{\mathbf{A}^\top t_{i+j}} \mathbf{N}^\top \hat{\mathbf{Q}}_{i-1,j} \quad (\text{for } i > 1) \\
\mathbf{Q}_{i,j} &= \int_0^\infty \cdots \int_0^\infty \hat{\mathbf{Q}}_{i,j} \hat{\mathbf{Q}}_{i,j}^\top dt_1 \ldots dt_{i+j}
\end{aligned}$$

Then the observability Gramian $\mathbf{Q}$ is given by $\boxed{\mathbf{Q} = \mathbf{Q}^{\mathsf{bil}} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{Q}_{i,j}.}$

- Each $\mathbf{Q}_{i,j}$ component satisfies its own matrix equation. For example,

$$\mathbf{A}^\top \mathbf{Q}_{i,j} + \mathbf{Q}_{i,j} \mathbf{A} + \mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} = 0 \quad (\text{for } i > 1)$$

Gramians and Lyapunov equations

Theorem ([P./Gosea/Pontes Duff/Benner '24])

The observability Gramian $\mathbf{Q}$ for the BQO system $\mathbf{\Sigma} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{N}, \mathbf{M})$ satisfies the following linear matrix equation,

$$\mathbf{A}^\top \mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{M}\mathbf{P}\mathbf{M} + \mathbf{N}^\top \mathbf{Q}\mathbf{N} + \mathbf{C}^\top \mathbf{C} = 0, \tag{3}$$

where $\mathbf{P}$ is the reachability Gramian for BQO systems which satisfies,

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{N}\mathbf{P}\mathbf{N}^\top + \mathbf{B}\mathbf{B}^\top = 0. \tag{4}$$

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- $\mathbf{Q}$ can be derived by computing the **reachability Gramian for the dual system** similar to [Benner/Goyal '24].
- It can be shown that the nullspace of $\mathbf{Q}$ corresponds to the unobservable subspace of the BQO system.

Part - II: Model order reduction

Balanced Truncation

- Originally introduced in:

  Principal component analysis in linear systems: Controllability, observability, and model reduction, B. Moore, IEEE Transactions on Automatic Control, 26 (1), 1981.

  and

  Model reduction via balanced state space representations , L. Pernebo, L. Silverman, IEEE Transactions on Automatic Control, 27 (2), 382-387, 1982.

Essence of Balanced Truncation

- Find $\mathbf{T}$ (balancing transform) such that the Gramians of the transformed system are equal and diagonal (balanced).

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- Now consider the balanced $(\mathbf{P} = \mathbf{Q})$ system:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix},$$
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- The subsystem $\boldsymbol{\Sigma}_r^{\mathsf{BQO}} = (\mathbf{A}_{11}, \mathbf{B}_1, \mathbf{C}_1, \mathbf{N}_{11}, \mathbf{M}_{11})$ is the reduced order model of $\boldsymbol{\Sigma}^{\mathsf{BQO}}$, where $\mathbf{A}_{11} \in \mathbb{R}^{r \times r}$.

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- It can be shown that $\mathbf{S}_1$ is a generalized Gramian of the reduced BQO system, i.e., it satisfies the following inequality,

$$\mathbf{A}_{11}^\top \mathbf{S}_1 + \mathbf{S}_1 \mathbf{A}_{11} + \mathbf{M}_{11} \mathbf{S}_1 \mathbf{M}_{11} + \mathbf{C}_1^\top \mathbf{C}_1 + \mathbf{N}_{11}^\top \mathbf{S}_1 \mathbf{N}_{11} \leq 0.$$

Balanced truncation

- Balanced truncation for Quadratic bilinear (QB) systems and linear systems with quadratic output (LQO) have been shown in [Benner/Goyal '24] and [Benner/Goyal/Pontes Duff '20].

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- We follow a similar recipe to obtain the balanced truncation counterpart for BQO systems.
- We have already seen how to obtain $\mathbf{P}, \mathbf{Q}$, i.e., by solving matrix equations.
- The square root method has the advantage over the 'naive' method in that we do not have to compute the balancing transform matrix which tends to be ill-conditioned.

Square-root BT algorithm

- Let $\mathbf{U}, \mathbf{L}$ be square factors of the Gramians, i.e., $\mathbf{P} = \mathbf{U}\mathbf{U}^\top$ and $\mathbf{Q} = \mathbf{L}\mathbf{L}^\top$.

Square-root BT algorithm

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- Compute the truncated **SVD** of matrix $\mathbf{L}^\top\mathbf{U}$:

$$\mathbf{L}^\top\mathbf{U} = \begin{bmatrix}\mathbf{Z}_1 & \mathbf{Z}_2\end{bmatrix} \begin{bmatrix}\mathbf{S}_1 & \\ & \mathbf{S}_2\end{bmatrix} \begin{bmatrix}\mathbf{Y}_1^\top \\ \mathbf{Y}_2^\top\end{bmatrix}, \quad \mathbf{S}_1 \in \mathbb{R}^{r\times r}$$

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- Construct the model reduction bases

$$\mathbf{W}_r = \mathbf{L}\mathbf{Z}_1\mathbf{S}_1^{-1/2} \text{ and } \mathbf{V}_r = \mathbf{U}\mathbf{Y}_1\mathbf{S}_1^{-1/2}.$$

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- The matrices of reduced-order balanced system $\mathbf{\Sigma}_r$ are given by

$$\begin{aligned}
\mathbf{A}_r &= \mathbf{W}_r^\top\mathbf{A}\mathbf{V}_r = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^\top\left(\mathbf{L}^\top\mathbf{A}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},\\
\mathbf{B}_r &= \mathbf{W}_r^\top\mathbf{B} = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^\top\left(\mathbf{L}^\top\mathbf{B}\right),\\
\mathbf{C}_r &= \mathbf{C}\mathbf{V}_r = \left(\mathbf{C}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},\\
\mathbf{N}_r &= \mathbf{W}_r^\top\mathbf{N}\mathbf{V}_r = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^\top\left(\mathbf{L}^\top\mathbf{N}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},\\
\mathbf{M}_r &= \mathbf{V}_r^\top\mathbf{M}\mathbf{V}_r = \mathbf{S}_1^{-1/2}\mathbf{Y}_1^\top\left(\mathbf{U}^\top\mathbf{M}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2}.
\end{aligned}$$

Matrix equations for Gramians

- The matrix equations for $\mathbf{P}, \mathbf{Q}$ are **not** standard, but generalized Lyapunov equations, due to the $\mathbf{N}\mathbf{P}\mathbf{N}^\top$ and $\mathbf{N}^\top\mathbf{Q}\mathbf{N}$ term respectively.

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- Each of the components for $\mathbf{Q}$ have specific matrix equations.
- Consider $\mathbf{Q}_{2,1}$ for example, e.g.,

$$\mathbf{A}^\top\mathbf{Q}_{2,1} + \mathbf{Q}_{2,1}\mathbf{A} + \mathbf{N}^\top\mathbf{Q}_{1,1}\mathbf{N} = 0$$

The equation for $\mathbf{Q}_{2,1}$ becomes a Lyapunov one, once $\mathbf{Q}_{1,1}$ is known!

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Observation

In theory, we could solve infinite Lyapunov equations to get $\mathbf{Q}$. Alternatively, we can approximate $\mathbf{Q}$ by only solving the first few significant ones! **(Method 2)**

Truncated Gramians

Definition

We define the truncated Gramians $\mathbf{P}_\tau$ *and* $\mathbf{Q}_\tau$ *as,*

$$\mathbf{P}_\tau = \mathbf{P}_1 + \mathbf{P}_2 \quad \textit{(see [Benner/Goyal/Redmann '17])}$$
$$\mathbf{Q}_\tau = \mathbf{Q}_1^{bil} + \mathbf{Q}_2^{bil} + \mathbf{Q}_{1,1} + \mathbf{Q}_{1,2} + \mathbf{Q}_{2,1} + \mathbf{Q}_{2,2}.$$

where the RHS terms are the first few components of $\mathbf{P}, \mathbf{Q}$ *respectively.*

- We solve for the terms - $\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1^{\text{bil}}, \mathbf{Q}_2^{\text{bil}}, \mathbf{Q}_{1,1}, \mathbf{Q}_{1,2}$ in that order to ensure that we only need to solve Lyapunov equations.

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- Now, by replacing $\mathbf{P}, \mathbf{Q}$ with $\mathbf{P}_\tau, \mathbf{Q}_\tau$ in the square root BT algorithm, we can obtain our reduced matrices.
- More truncation terms imply a better approximation of $\mathbf{P}, \mathbf{Q}$!

Comparison of the two methods

Method 1

- Solve:
$\mathbf{AP} + \mathbf{PA}^\top + \mathbf{NPN}^\top + \mathbf{BB}^\top = 0$
$\mathbf{A}^\top\mathbf{Q} + \mathbf{QA} + \mathbf{MPM} + \mathbf{N}^\top\mathbf{QN} + \mathbf{C}^\top\mathbf{C} = 0.$

Method 2

- Solve 6 Lyapunov equations corresponding to - $\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1^{\mathsf{bil}}, \mathbf{Q}_2^{\mathsf{bil}}, \mathbf{Q}_{1,1}, \mathbf{Q}_{1,2}$, respectively.

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- ADI solvers

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- MATLAB - `lyapchol`

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- ADI solvers
- Expensive

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- Use full Gramians $(\mathbf{P},\mathbf{Q})$ to compute projection matrices

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- Balanced truncation with full Gramians - BT

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- MATLAB - `lyapchol`
- Cheap to compute
- Use truncated Gramians $(\mathbf{P}_\tau, \mathbf{Q}_\tau)$ to compute projection matrices
- Balanced truncation with truncated Gramians - truncBT

Part - III: Numerical Results

Heat transfer model

Heat equation in a square $[0, 1]^2$ [Benner/Damm '11]

$$\boxed{\begin{aligned} \partial_t x &= \Delta x, && \text{in } (0,1) \times (0,1), \\ n \cdot \nabla x &= 0.8 u_1 x && \text{on } \Gamma_1, \\ x &= u_2, && \text{on } \Gamma_2, \\ x &= 0, && \text{on } \Gamma_3, \Gamma_4, \end{aligned}}$$

- The boundary Γ_1 has a bilinear term.

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- The boundary Γ_1 has a bilinear term.
- A semi-discretization in space using finite differences with $k = 15$ grid points results in a bilinear system of dimension $n = 225$ of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = 0.$$

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- We choose $\mathbf{M}$ to be a tridiag$(-0.33, 0.67, -0.33)$ matrix with entries and $\mathbf{C}$ as a row vector with each entry $= 0.0044$.

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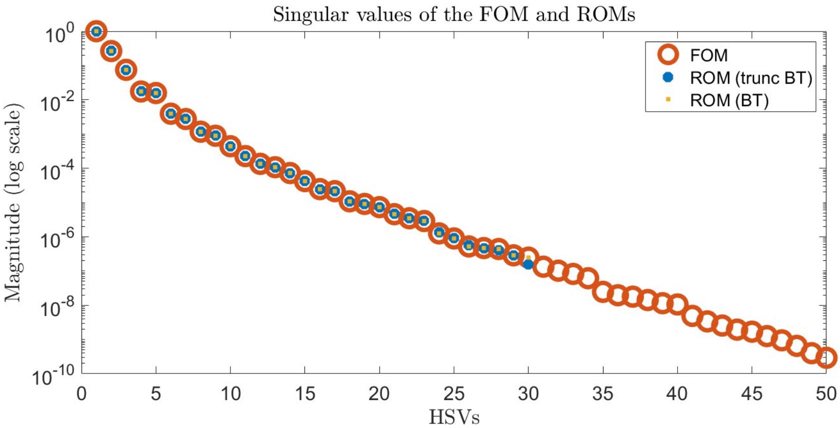
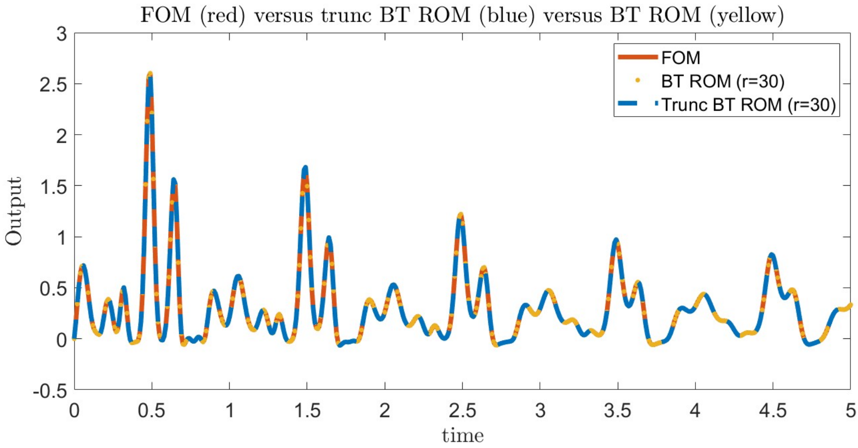
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = 0.$$

- We choose $\mathbf{M}$ to be a tridiag$(-0.33, 0.67, -0.33)$ matrix with entries and $\mathbf{C}$ as a row vector with each entry $= 0.0044$.
- Construct reduced systems of order $r = 30$ for both `BT` and `TruncBT`.

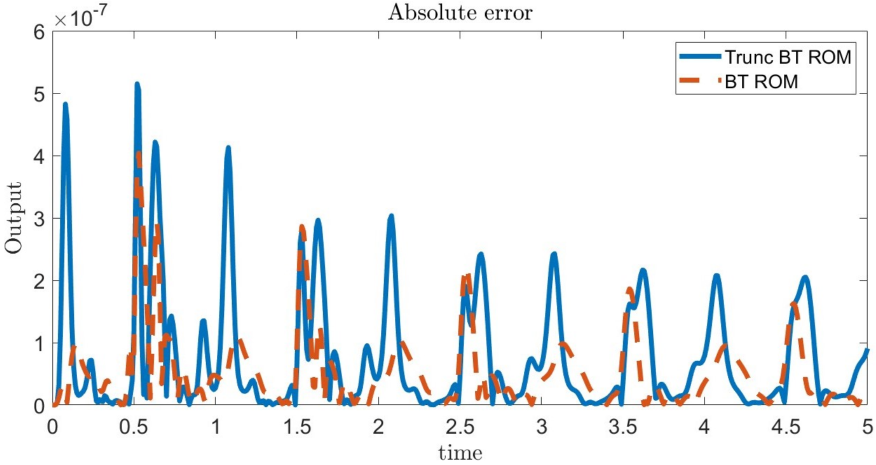
Heat transfer model (singular values)



Heat transfer model (time evolution)



Heat transfer model (absolute error)



Conclusions and outlook

Key takeaways

1. Define **BQO kernels** using the Volterra series representation;
2. Taylor the **BQO observability Gramian** and express it as the solution of (linear) matrix equations;
3. Extend **Balanced truncation** to BQO systems;
4. Propose **truncated Gramians** as an effective and 'cheap' alternative to using the full Gramians.

Future directions

1. Relating Gramians with energy functionals **(work in progress)**;
2. How do the **neglected singular values** relate to the error between the full and reduced order models?

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